Theoretical Physics IV (Statistical Mechanics)

Sample Exam Duration: 150 min – Max. Points: 100

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1. Thermodynamics of a gas (8+10 Points = 18 Points)

The internal energy of a gas of N particles in a volume V is given by:

$$E(S,V,N) = a\frac{S^2}{N} + b\frac{N^3}{V^2}.$$

(a) Express the temperature T, the pressure P and the chemical potential μ of the gas as functions of S, V und N.

(b) Evaluate the specific heat at constant volume C_V as function of T, V and N as well as the isothermal compressibility $\kappa_T = -1/V\,(\partial V/\partial P)_{T,N}$ as function of T, P and N. Under which conditions for a and b is the gas thermally and mechanically stable?

2. Interacting spins (20+8 Points = 28 Points)

Consider two interacting Ising-spins σ_1, σ_2 in a magnetic field, whose energy is given by

$$E = -J\,\sigma_1\,\sigma_2 - \mu\,B\,\big(\sigma_1 + \sigma_2\big), \quad \sigma_1, \sigma_2 = \pm 1/2, \quad J > 0.$$

(a) Assuming $B = 0$, evaluate the canonical partition function Z, the free energy F and the specific heat C as functions of the temperature T. Give the behavior of C at low and high temperatures and sketch $C(T)$.

(b) Evaluate the canonical partition function Z, the free energy F and the magnetization M, now for the case of finite magnetic field $B \neq 0$.

3. Pauli susceptibility (1+2+4+4+6+5 Points = 22 Points)

Consider a gas of N free fermions in a volume $V = L^3$. The particles have spin 1/2 and their single-particle dispersion relation in magnetic field is given by $\epsilon_\sigma(\vec{p}) = \vec{p}^2/2m - B\sigma$, with $\sigma = \pm 1$. The single-particle states are labeled by the quantum numbers $\lambda = (\vec{p}, \sigma) = (p_x, p_y, p_z, \sigma)$.

(a) Which are the possible values for the occupation number n_λ of a single-particle state λ?

(b) Express the energy E_α and the number of particles N_α of a many-body state $|\alpha\rangle$ in terms of the occupation number n_λ introduced above.

(c) Prove that the grand canonical distribution function Z of the system is of the form $Z = \prod_\lambda Z_\lambda$ and evaluate Z_λ.

(d) Prove that for $B = 0$ and large V, the density of states $D(\epsilon)$ per spin and unit volume is proportional to the square-root of energy, $D(\epsilon) \propto \sqrt{\epsilon}$.

(e) Using $M = -(\partial J/\partial B)_{T,V,\mu}$, with J the grand canonical potential, show that the magnetization M can be expressed as:

$$M = V\int_0^\infty d\epsilon\, D(\epsilon)\big[n(\epsilon-\mu-B) - n(\epsilon-\mu+B)\big], \qquad n(x) = \frac{1}{1+e^{x/k_BT}}.$$

(f) Give the magnetic susceptibility χ at zero temperature and zero magnetic field ($B=0$ and $T=0$) as a function of the density of states at the Fermi energy.

4. Classical gas of hard sphere in 1D (10+6 Points = 16 Points)

Consider N hard (impenetrable) spheres with radius r, which can move freely in the interval $-r \le x \le L+r$ of the x-axis. The classical Hamiltonian of this system reads

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + V(x_1, x_2, \ldots, x_N),$$

with

$$V(x_1, x_2, \ldots, x_N) = \begin{cases} \infty, & \text{if there are two sphere } i,j \text{ such that } |x_i - x_j| \le 2r, \\ \infty, & \text{if there is one sphere } i \text{ such that } x_i < 0 \text{ or } x_i > L, \\ 0, & \text{otherwise,} \end{cases}$$

where x_i is the position of the center of the i-th sphere.

(a) Prove that the classical canonical partition function is given by

$$Z = \frac{\big(L - 2r(N-1)\big)^N}{\lambda_T^N\, N!}, \qquad \lambda_T = \sqrt{\frac{2\pi\hbar^2}{mk_BT}}.$$

The particle-number dependent prefactor in Z is $1/(2\pi\hbar)^N$ (no factor $1/N!$).

(b) Evaluate the free energy F and the pression $P = -(\partial F/\partial L)_{T,N}$ of the gas, and interpret the obtained equation of state.

Hint: $\int_{-\infty}^{\infty} dy\, e^{-\alpha y^2} = \sqrt{\pi/\alpha}$.

5. Comprehension test (2+2+2+2+2+2+4 Points = 16 Points)

Answer the following questions with either a short sentence, a formula or a short calculation.

(a) State one formulation of the second law of thermodynamics.

(b) What is a perpetual motion machine of the second kind?

(c) Consider a quantum system with a two-fold degenerate ground state. Which is the entropy at $T=0$?

(d) State the Bose-Einstein distribution $n(\epsilon)$.

(e) Give the temperature dependence of the magnetic susceptibility of a free spin.

(f) Consider a quantum system with a finite energy gap Δ above the ground state. Give the temperature dependance of the specific heat at low temperatures.

(g) State the equation of state and the internal energy (as a function of number of particle and temperature) of a classical, ideal, monoatomic gas.